GP Subject Math - Linear Albegra Handout

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The following handout's goal is to cover the most important facts and results in Linear Algebra. The full details can be found in the linear algebra module.

Dimension of Ker and Im:

Let V, W be vector spaces over $\mathbb{R}$ or $\mathbb{C}$, and Let $T : V \to W$ be a linear transformation.
Theorem 1. $\dim(\text{Ker}T) + \dim(\text{Im}T) = \dim V$

The meaning of this theorem is that the transformation T takes the sapce V and "divides" it into two parts, one part is sent to 0 (Ker) and the other part is sent to non zero elements in W constructing the subspace $\text{Im}T$.

Thinking of T as an $m \times n$ matrix ($n = \dim V, m = \dim W$), we have the following equivalent result(denote $r = \text{Rank}(A)$):
Theorem 2. $\dim(\text{Ker}A) + r = n \Rightarrow \dim(\text{Ker}A) = n - r$

Dimension of sum of subspaces:

Let U, W be subspaces of V. Then $U \cap W$ and $U + W$ are also subspaces of V. and we have:
Theorem 3. $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$*, and all the terms are bounded above by* $\dim V$*.*

It is like we are counting the dimension of the intersection twice in the sum.

Composition of transformations

When $W = V$, we get $T : V \to V$. Consider another transformation $L : V \to V$. Now we can compose $T \circ L : V \to V$ or $L \circ T : V \to V$ (not necessarily the same, equivalent to matrix multiplication).

Notice that if $v \in \text{Ker}L \Rightarrow v \in \text{Ker}TL$ since $TL(v) = T(0) = 0$. So the kernel of TL contains the kernel of L. **Composition makes the kernel "bigger"**. Thinking about matrices, we conclude that the null space of AB is bigger than the null Space of just B. So $\text{Rank}AB \leq \text{Rank}B$.

Also, take $B = A$ so $\text{Rank}A^2 \leq \text{Rank}A \Rightarrow \text{Rank}A^n \leq \text{Rank}A$ for every $n \in \mathbb{N}$.

Eigenvalues

Theorem 4. *For a given $n \times n$ matrix A (or a linear transformation), the sum of its eigenvalues is* trace(A) *and the product of its eigenvalues is* Det$A$. *Furthermore, the characteristic polynomial(char.poly) is* $x^n - \text{trace}(A)x^{n-1} + (things - without - a - short - formula) + (-1)^n \det A$.

Special and important case: For 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the char-poly should be calculated in seconds! $x^2 - (a+d)x + (ad - bc)$.

Theorem 5. *Let $\lambda_1, \lambda_2, ..., \lambda_l$ be the eigenvalues of A. Let $B = \alpha A + \beta I$ where $\alpha, \beta \in \mathbb{C}$, I is the identity matrix. Then the eigenvalues of B are $\alpha\lambda_1 + \beta, \alpha\lambda_2 + \beta, ..., \alpha\lambda_l + \beta$ with the same corresponding eigenvectors.*

The latter gives us a quick formula to calculate eigenvalues of complicated matrices using simple matrices.

The eigenvalues of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are 0 and 2. So the eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $0-1, 2-1 = -1, 1$.

Sum of rows trick If the sum of the entries in each row is the same, then this is an eigenvalue with an eigenvector:

$$v = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

For example the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 6 & 0 \\ i & 3 & 3-i \end{pmatrix}$$

has an eigenvalue of 6 with eigenvector:

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Suppose the question is "what are the eigenvalues of A"? We can eliminate all the possibilities without 6 (and the other two must sum up to $4 - i$ due to the trace).

Invertability

Theorem 6. *The following are equivalent for an $n \times n$ matrix A or the corresponding $T : V \to V$:*

(a) *A is invertible*

(b) *A^T (transpose) is invertible*

(c) $\text{Det}A \neq 0$.

(d) *$RankA = n$.*

(e) $\dim \ker A = 0$.

(f) *$Ax = 0$ has only the trivial solution.*

(g) *$Ax = b$ has a single solution for any b with the right dimensions.*

(h) *A does not have 0 as an eigenvalue.*

(i) *The columns of A form a basis for $\mathbb{R}^n$ or $\mathbb{C}^n$*

Similarity

Theorem 7. *When A, B are similar matrices, they have the same: eigenvalues, determinant, trace, char.poly.*

By definition of similarity, exists an invertible matrix P s.t. $P^{-1}AP = B$. We can manipulate the latter formula to conclude that A^{-1} and B^{-1} are similar, and also $\alpha A + \beta I$ and $\alpha B + \beta I$ are similar. Note that "similarity" is an equivalence relation.

Trace and Det

- $\text{trace} : \mathbb{R}^{n \times n} \to \mathbb{R}$ is a linear transformation.
  - $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$
  - $\text{trace}(cA) = c\,\text{trace}(A)$
  - $\text{trace}(A^T) = \text{trace}(A)$
- The Determinant is not linear! (i.e., $\det(A + B) \neq \det(A) + \det(B)$ and $\det(cA) \neq c\det(A)$ in general)

Important Examples of Linear Transformations

Let $T : V \to U$ be a linear transformation.

- If $\dim(V) = \dim(U)$, then T onto $\iff$ T one-to-one $\iff$ T invertible.
- Note that this is not necessarily true if $\dim(V) \neq \dim(U)$.
- Example: The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ with $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = a_1$ is onto but not one-to-one.
- Example: The linear transformation $T : \mathbb{R} \to \mathbb{R}^2$ with $T(a) = \begin{bmatrix} a \\ 0 \end{bmatrix}$ is one-to-one but not onto.

A few more remarks

- The collection of all $m \times n$ matrices is a vector space of dimension mn.
- Make sure to practice row reducing and determinant calculation. It may be time consuming.
- A real symmetric matrix is always diagonalizable and has real eigenvalues.
- Usually the char.poly contains rational coefficients. So we can use the methods from the Basic module to guess all the rational eigenvalues.

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